7.2 - Inverse Transforms and Transforms of Derivatives

Theorem 7.2.1: Some Inverse Transforms

$$\text{(a) } 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

(b) $t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$, $n = 1,\ 2,\ 3,\ \dots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$

Example: Find the inverse Laplace transform.

$\frac{1}{6}\mathscr{L}^{-1}\left\{\frac{16}{s^4}\right\}$ $\quad n=3$

$\frac{1}{3}\int 3e^{3x}\,dx$

$\frac{1}{3}e^{3x}+C$

$= \frac{1}{6}t^3$

$\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$ cosine

$= 10\cos 4t$

$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{1}{\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{1\cdot\sqrt{2}}{s^2+2}\right\}$

$= \boxed{\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t}$

$\int \frac{x+1}{x^2-4x}\,dx$ PFD

$s+1 = (A+B)s - 4A$

$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

$s(s-4)$

$$\frac{s+1}{s(s-4)} = \frac{A}{s} + \frac{B}{s-4} \rightarrow s+1 = A(s-4)+Bs$$

constant

linear

$s=0$: $A = -\frac{1}{4}$

$s=4$: $B = \frac{5}{4}$

$$\mathscr{L}^{-1}\left\{-\frac{1/4}{s} + \frac{5/4}{s-4}\right\}$$

$$= \boxed{-\frac{1}{4} + \frac{5}{4}e^{4t}}$$

Example: Use the Laplace transform to solve the Initial-Value Problem.

$$2\frac{dy}{dt} + y = 0, \qquad y(0) = -3$$

Basic plan

$y(t) \xrightarrow{\mathscr{L}} Y(s)$

$Y(s) \xrightarrow{\mathscr{L}^{-1}} y(t)$

Plan: Apply the Laplace transform to both sides: $2\mathscr{L}\left\{\frac{dy}{dt}\right\} + \mathscr{L}\{y\} = 0$

Transforms of derivatives

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$u = e^{-st}$ $\quad dv = f'(t)dt$

$du = -se^{-st}dt$ $\quad v = f(t)$

$$= \left.\frac{f(t)}{e^{st}}\right|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

0

$$= -f(0) + sF(s)$$

$$\Rightarrow \mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

from initial condition

plan, cont: solve for $Y(s)$ and use the inverse Laplace transform to find y

$$\underline{2\frac{dy}{dt}} + \underline{y} = \underline{0}, \quad y(0) = -3$$

$$\underline{2(sY(s)+3)} + \underline{Y(s)} = \underline{0}$$

$$2sY(s) + Y(s) = -6 \Rightarrow Y(s) = -\frac{6}{2s+1}$$

$$Y(s) = -\frac{3}{s+1/2} \overset{\mathcal{L}^{-1}}{\Rightarrow} \boxed{y(t) = -3e^{-1/2 t}}$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

$u = e^{-st}$ $\quad dv = f''(t)\,dt$

$du = -s e^{-st} dt$ $\quad v = f'(t)$

$$= \frac{f'(t)}{e^{st}}\Big|_0^\infty + s\boxed{\int_0^\infty e^{-st} f'(t)\,dt}$$

$$= -f'(0) + s\,\mathcal{L}\{f'(t)\}$$

$$= -f'(0) + s[sF(s) - f(0)]$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Recall $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Guess $\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - sf'(0) - f''(0)$

Theorem: Transform of a Derivative

If $f,\ f',\ \ldots,\ f^{(n-1)}$ are continuous on $[0,\ \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0,\ \infty)$, then

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

Example: Use the Laplace transform to solve the Initial-Value Problem.

$y'' - 4y' = 6e^{3t} - 3e^{-t}, \qquad y(0) = 1, \quad y'(0) = -1$

$-4(sY(s) - y(0))$

$$s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) = \frac{6}{s-3} - \frac{3}{s+1}$$

$$(s^2 - 4s)Y(s) - s + 1 + 4 = \frac{6}{s-3} - \frac{3}{s+1}$$

$$s(s-4)Y(s) = \frac{6}{s-3} - \frac{3}{s+1} + s - 5$$

$6s + 6 - 3s + 9 + \quad (s+1)(s^2 - 8s + 15)$

$$s(s-4)Y(s) = \frac{6(s+1) - 3(s-3) + (s-5)(s-3)(s+1)}{(s-3)(s+1)}$$

$$s(s-4)Y(s) = \frac{3s + 15 + s^3 - 8s^2 + 15s + s^2 - 8s + 15}{(s-3)(s+1)}$$

$$Y(s) = \frac{s^3 - 7s^2 + 10s + 30}{s(s-4)(s-3)(s+1)}$$

$$\frac{s^3 - 7s^2 + 10s + 30}{s(s-4)(s-3)(s+1)} = \frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-3} + \frac{D}{s+1}$$

$$s^3 - 7s^2 + 10s + 30 = A(s-4)(s-3)(s+1) + Bs(s-3)(s+1)$$
$$+ Cs(s-4)(s+1) + Ds(s-4)(s-3)$$

$s = 0 \qquad 30 = 12A \Rightarrow A = 5/2$

$s=4$ $\quad 64-112+40+30=20B \Rightarrow B=\frac{11}{10}$

We similarly find $C=-2$, $D=-3/5$

Wolfram alpha.com: Apart((numerator)/(denom))

$$Y(s)=\frac{5/2}{s}+\frac{11/10}{s-4}-\frac{2}{s-3}-\frac{3/5}{s+1}$$

$$\boxed{y(t)=\frac{5}{2}+\frac{11}{10}e^{4t}-2e^{3t}-\frac{3}{5}e^{-t}}$$